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GEOMETRY OF COMPLEX MANIFOLDS SIMILAR TO THE CALABI-ECKMANN MANIFOLDS

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In [4] Calabi and Eckmann showed that the product of two odd-dimensional spheres $S^{2p+1} \times S^{2q+1}$ $(p, q \geq 1)$ is a complex manifold. As $S^{2p+1} \times S^{2q+1}$ is not Kaehlerian, the fundamental 2-form Ω of the Hermitian structure is not closed. However, $d\Omega$ does have a special form on $S^{2p+1} \times S^{2q+1}$; in fact, $S^{2p+1} \times S^{2q+1}$ admits two nonvanishing vector fields which are both Killing and analytic, and whose covariant forms determine Ω. Our purpose here is to study complex manifolds whose complex structures are similar to the complex structure on $S^{2p+1} \times S^{2q+1}$.

In § 1 we review the geometry of the Calabi-Eckmann manifolds. In § 2 we give some elementary properties of vector fields on a Hermitian manifold, and introduce the notion of a holomorphic pair of automorphisms and of a bicontact manifold. § 3 continues the author's paper [2] on the differential geometry of principal toroidal bundles for the present case. In § 4 we discuss bicontact manifolds and, in particular, the integrable distributions of a bicontact structure on a Hermitian manifold. Finally in § 5 we give some results on the curvatures of a Hermitian manifold admitting a holomorphic pair of automorphisms.

1. The Hermitian structure on the Calabi-Eckmann manifolds

The construction of the complex structure on $S^{2p+1} \times S^{2q+1}$ which we will give is due to Morimoto [6]. It is well known that an odd-dimensional sphere S^{2p+1} carries a contact structure, i.e., a nonvanishing 1-form η such that $\eta \wedge (d\eta)^p \neq 0$. Let G be the standard metric on S^{2p+1}. Then there exist on S^{2p+1} (see e.g. [8]) a contact form η, a vector field ξ, and a tensor field φ of type $(1, 1)$ such that

$$\eta(\xi) = 1 \,, \quad \varphi\xi = 0 \,, \quad \eta \circ \varphi = 0 \,, \quad \varphi^2 = -I + \eta \otimes \xi \,,$$
$$G(\xi, X) = \eta(X) \,, \quad G(\varphi X, \varphi Y) = G(X, Y) - \eta(X)\eta(Y) \,,$$

i.e., S^{2p+1} carries an almost contact metric structure. For a contact structure $\eta \wedge (d\eta)^p \neq 0$, φ, ξ and G may be chosen such that $d\eta(X, Y) = G(\varphi X, Y)$,

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as happens in the sphere example. Moreover, the contact metric structure on S^{2p+1} is normal, i.e.,

$$[\varphi, \varphi] + d\eta \otimes \xi = 0 ,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. Thus S^{2p+1} carries a normal contact metric or *Sasakian* structure.

Now let (φ, ξ, η, G) and $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{G})$ be Sasakian structures on S^{2p+1} and S^{2q+1} respectively. Then define an almost complex structure J on $S^{2p+1} \times S^{2q+1}$ by

$$J(X, \bar{X}) = (\varphi X - \bar{\eta}(\bar{X})\xi, \bar{\varphi}\bar{X} + \eta(X)\bar{\xi}) ,$$

and let g be the product metric. Then direct computations show [6] that $J^2 = -I$, $g(J(X, \bar{X}), J(Y, \bar{Y})) = g((X, \bar{X}), (Y, \bar{Y}))$ and, using normality, that $[J, J] = 0$. Thus $S^{2p+1} \times S^{2q+1}$ is a Hermitian manifold.

Defining the fundamental 2-form Ω of the Hermitian structure by

$$\Omega((X, \bar{X}), (Y, \bar{Y})) = g(J(X, \bar{X}), (Y, \bar{Y})) ,$$

we find that

$$\Omega = d\eta + d\bar{\eta} + \eta \wedge \bar{\eta} ,$$

where we view η and $\bar{\eta}$ as 1-forms extended to the product. Thus the fundamental 2-form Ω of the Hermitian structure on $S^{2p+1} \times S^{2q+1}$ satisfies

$$d\Omega = d\eta \wedge \bar{\eta} - \eta \wedge d\bar{\eta} .$$

Finally we remark that from the Hopf fibration $\pi' \colon S^{2p+1} \to PC^p$ of an odd-dimensional sphere as a principal circle bundle over complex projective space, we obtain a natural fibration $\pi \colon S^{2p+1} \times S^{2q+1} \to PC^p \times PC^q$ of a Calabi-Eckmann manifold as a principal T^2 (2-dimensional torus) bundle over a product of complex projective spaces. In fact the complex coordinates of $S^{2p+1} \times S^{2q+1}$ are essentially those of $PC^p \times PC^q$ together with a fibre coordinate [4], [5].

2. Some elementary properties of vector fields on a Hermitian manifold

Let M^{2n} be a Hermitian manifold with complex structure J and Hermitian metric g. Let U be an analytic vector field[1] on M^{2n}, i.e., $\mathfrak{L}_U J = 0$ where $\mathfrak{L}$ denotes Lie differentiation.

[1] More generally on an almost complex manifold a vector field U is said to be almost analytic if $\mathfrak{L}_U J = 0$ and $[J, J](U, X) = 0$ for all vector fields X.

Proposition 2.1. *If U is an analytic vector field on M^{2n}, then so is $V = JU$.*
Proof.

$$\begin{aligned} 0 = [J,J](U,X) &= -[U,X] + [V,JX] - J[V,X] - J[U,JX] \\ &= -J(\mathfrak{L}_U J)X + (\mathfrak{L}_V J)X = (\mathfrak{L}_V J)X\,. \end{aligned}$$

Thus, if U is an infinitesimal automorphism of J, so is JU; but if U is Killing (an automorphism of g), JU is not in general Killing. We therefore make the following definition.

Definition. By a holomorphic pair of automorphisms we mean a unit vector field U such that U and $V = JU$ are infinitesimal automorphisms of the Hermitian structure.

Let u and v denote the covariant forms of U and V respectively. We begin with some elementary properties of a holomorphic pair of automorphisms $(U, V = JU)$.

Lemma 2.2. $[U, V] = 0$.

Proof. $0 = (\mathfrak{L}_U J)U = [U, JU] - J[U, U] = [U, V]$.

Lemma 2.3. $du(U,X) = 0$, $du(V,X) = 0$, $dv(U,X) = 0$, $dv(V,X) = 0$.

Proof. We give the proof for du, the proof for dv being similar. Since U is Killing and unit, we have

$$\begin{aligned} du(U,X) &= (\nabla_U u)(X) - (\nabla_X u)(U) = g(\nabla_U U, X) - g(\nabla_X U, U) \\ &= -2g(\nabla_X U, U) = 0\,, \end{aligned}$$

where ∇ denotes the Riemannian connection of g. Similarly since $[U, V] = 0$ and V is also Killing, we have

$$du(V,X) = g(\nabla_V U, X) - g(\nabla_X U, V) = g(\nabla_U V, X) + g(\nabla_X V, U) = 0\,.$$

Proposition 2.4. *At each point of M^{2n}, u and v have odd rank, i.e., there exist nonnegative integers p and q such that $u \wedge (du)^p \neq 0$, $v \wedge (dv)^q \neq 0$, $(du)^{p+1} = 0$, $(dv)^{q+1} = 0$.*

Proof. First note that $(du)^n = 0$; for evaluating $(du)^n$ on a J-basis containing U and V each term in

$$(du)^n(U, V, X_3, \cdots, X_{2n})$$

vanishes by Lemma 2.3; here we have set $X_1 = U$, $X_2 = JU = V$ and $\{X_i\}$ a J-basis. Suppose now that at $m \in M^{2n}$, $(du)^p \neq 0$ and $(du)^{p+1} = 0$. Then evaluating $(u \wedge (du)^p)(U, Y_1, \cdots, Y_{2p})$ where $Y_1, \cdots, Y_{2p}$ are vector fields such that $du(Y_i, Y_j) \neq 0$, we have that $u \wedge (du)^p \neq 0$. Similarly v has rank $2q + 1$.

Definition. We say that a differentiable manifold M^{2n} is bicontact if it admits 1-forms u and v such that $u \wedge v \wedge (du)^p \wedge (dv)^q \neq 0$, $(du)^{p+1} = 0$

and $(dv)^{q+1} = 0$ with $p + q + 1 = n$. M^{2n} is called a Hermitian bicontact manifold if M^{2n} is both Hermitian and bicontact, and the 1-forms u and v are the covariant forms of a holomorphic pair of automorphisms.

Lemma 2.5. *If du is of bidegree $(1, 1)$ with respect to the complex structure J, then so is dv.*

Proof. Recall that the Nijenhuis torsion of a vector-valued 1-form h is given by its action on a 1-form θ. This action is

$$[h, h]\theta = -h^{(2)}d\theta + h^{(1)}d(\theta \circ h) - d(\theta \circ h^2) ,$$

where for a 2-form Θ,

$$(h^{(1)}\Theta)(X, Y) = \Theta(hX, Y) + \Theta(X, hY) , \qquad (h^{(2)}\Theta)(X, Y) = \Theta(hX, hY) .$$

$h^{(1)}\Theta$ is often denoted by $\Theta \barwedge h$. Now since $v = -u \circ J$ and du is of bidegree $(1, 1)$, we have

$$\begin{aligned} 0 &= ([J, J]u)(X, Y) \\ &= -du(JX, JY) - dv(JX, Y) - dv(X, JY) + du(X, Y) \\ &= -dv(JX, Y) - dv(X, JY) , \end{aligned}$$

and hence dv is of bidegree $(1, 1)$.

Remark. The above proof also shows that if $du = dv$, then $[J, J] = 0$ implies that $du(=dv)$ is of bidegree $(1, 1)$. The authors have studied certain manifolds admitting independent 1-forms u and v with $du = dv$, [1], [2].

Proposition 2.6. *If M^{2n} is Kaehlerian, then $du = dv = 0$.*

Proof. First since V is analytic, we have

$$0 = (\mathfrak{L}_V J)X = \nabla_V JX - \nabla_{JX}V - J\nabla_V X + J\nabla_X V = -\nabla_{JX}V + J\nabla_X V .$$

Now since V is Killing,

$$\begin{aligned} du(X, Y) &= g(\nabla_X U, Y) - g(\nabla_Y U, X) = g(-\nabla_X JV, Y) - g(-\nabla_Y JV, X) \\ &= g(\nabla_X V, JY) + g(J\nabla_Y V, X) = -g(\nabla_{JY}V, X) + g(J\nabla_Y V, X) = 0 . \end{aligned}$$

Similarly one can show that $dv = 0$.

In [9] one of the authors introduced the notion of an f-structure on a differentiable manifold, i.e., the manifold admits a tensor field $f \neq 0$ of type $(1, 1)$ satisfying $f^3 + f = 0$ (see also [1], [7]).

Proposition 2.7. *Let (M^{2n}, J, g) be an almost Hermitian manifold admitting a nonvanishing vector field U, then $U, V = JU, u, v$ (the covariant forms of U and V) and $f = J + v \otimes U - u \otimes V$ define an f-structure with complemented frames and rank $(f) = 2n - 2$ on M^{2n}, i.e., we have*

$$f^2 = -I + u \otimes U + v \otimes V\ , \quad fU = fV = 0\ , \quad u \circ f = v \circ f = 0\ ,$$
$$u(U) = v(V) = 1\ , \quad u(V) = v(U) = 0\ .$$

The proof of this proposition is a straightforward computation and will be omitted.

An f-structure with complemented frames (f, U, V, u, v) is said to be *normal* if the tensor S defined by

$$S(X, Y) = [f, f](X, Y) + du(X, Y)U + dv(X, Y)V$$

vanishes. Computing S in our case gives

$$S(X, Y) = [J, J](X, Y) - (du \barwedge J)(X, Y) - (dv \barwedge J)(X, Y)$$
$$+ u(X)(\mathfrak{L}_V J)Y - u(Y)(\mathfrak{L}_V J)X + v(X)(\mathfrak{L}_U J)Y - v(Y)(\mathfrak{L}_U J)X\ .$$

Thus we have the following result.

Proposition 2.8. *On a Hermitian manifold with a nonvanishing analytic vector field U, if du is of bidegree* (1, 1), *then the f-structure (f, U, V, u, v) is normal.*

It is well known (see e.g. [7]) that for a normal f-structure with complemented frames, we have

$$\mathfrak{L}_U f = 0\ , \quad \mathfrak{L}_U u = 0\ , \quad \mathfrak{L}_U v = 0\ , \quad \mathfrak{L}_V f = 0\ , \quad \mathfrak{L}_V u = 0\ , \quad \mathfrak{L}_V v = 0\ ,$$
$$du \barwedge f = 0\ , \quad dv \barwedge f = 0\ , \quad [U, V] = 0\ .$$

Thus a straightforward computation shows that $S = 0$ implies $[J, J] = 0$.

Now if g is the Hermitian metric on M^{2n}, then

$$g(fX, fY) = g(X, Y) - u(X)u(Y) - v(X)v(Y)\ ,$$
$$u(X) = g(U, X)\ , \qquad v(X) = g(V, X)\ ,$$

that is, (f, g, u, v) defines a metric f-structure with complemented frames.

Finally we define the fundamental 2-forms Ω and F of the structures by

$$\Omega(X, Y) = g(JX, Y)\ , \qquad F(X, Y) = g(fX, Y)\ .$$

Then a short computation gives

$$F = \Omega - u \wedge v\ .$$

3. Fibering by a holomorphic pair of automorphisms

In [2] the authors proved the following result.

Theorem. *Let M^{2m+s} be a compact connected manifold with a regular normal f-structure of rank $2m$. Then M^{2m+s} is the bundle space of a principal toroidal bundle over a complex manifold N^{2m}.*

Now if a complex manifold M^{2n} admits a regular analytic vector field U (i.e., every point $m \in M^{2n}$ has a neighborhood such that the integral curve of U through m passes through the neighborhood only once), the vector field $V = JU$ is not necessarily regular. Thus we say that a holomorphic pair of automorphisms is regular if both U and V are regular vector fields. Then using the above theorem and Proposition 2.8 we can prove the following result.

Theorem 3.1. *If a compact Hermitian manifold (M^{2n}, J, g) admits a regular holomorphic pair of automorphisms $(U, V = JU)$ with du of bidegree $(1, 1)$, then M^{2n} is a principal T^2 bundle over a Hermitian manifold N^{2n-2}.*

Proof. From the above theorem and Proposition 2.8 we obtain the desired fibration. Thus we shall only exhibit the Hermitian structure on N^{2n-2}. As U and V are analytic, J is projectable and we define J' on N^{2n-2} by

$$J'X = \pi_* J\tilde{\pi}X ,$$

where $\tilde{\pi}$ denotes the horizontal lift with respect to the Riemannian connection of g (in the nonmetric case one can use the pair (u, v) as a Lie algebra valued connection form to determine $\tilde{\pi}$ [2]). Then it is easy to check that $J'^2 = -I$. Moreover we have

$$\begin{aligned}[J', J'](X, Y) &= -[\pi_*\tilde{\pi}X, \pi_*\tilde{\pi}Y] + [\pi_*J\tilde{\pi}X, \pi_*J\tilde{\pi}Y] \\ &\quad -\pi_*J\tilde{\pi}[\pi_*J\tilde{\pi}X, \pi_*\tilde{\pi}Y] - \pi_*J\tilde{\pi}[\pi_*\tilde{\pi}X, \pi_*J\tilde{\pi}Y] \\ &= \pi_*[J, J](\tilde{\pi}X, \tilde{\pi}Y) = 0 .\end{aligned}$$

Finally as U and V are Killing, the metric g is projectable to a metric g' on N^{2n-2} given by $g'(X, Y) \circ \pi = g(\tilde{\pi}X, \tilde{\pi}Y)$. Then

$$g'(J'X, J'Y) \circ \pi = g(J\tilde{\pi}X, J\tilde{\pi}Y) = g(\tilde{\pi}X, \tilde{\pi}Y) = g'(X, Y) \circ \pi ,$$

and hence the structure on N^{2n-2} is Hermitian.

We now compute the fundamental 2-form F of the f-structure (f, U, V, u, v) on M^{2n}. First of all it is clear that $F(U, X) = 0$ and $F(V, X) = 0$. Thus it is enough to compute F on vector fields of the form $\tilde{\pi}X$, $\tilde{\pi}Y$ where X and Y are vector fields on N^{2n-2}.

$$\begin{aligned}F(\tilde{\pi}X, \tilde{\pi}Y) &= g(f\tilde{\pi}X, \tilde{\pi}Y) = g(J\tilde{\pi}X, \tilde{\pi}Y) = g(\tilde{\pi}J'X, \tilde{\pi}Y) \\ &= g'(J'X, Y) \circ \pi = \Omega'(X, Y) \circ \pi ,\end{aligned}$$

where Ω' is the fundamental 2-form on N^{2n-2}. Hence we have $F = \pi^*\Omega'$. Now $dF = d\pi^*\Omega' = \pi^*d\Omega'$ and $dF = d\Omega - du \wedge v + u \wedge dv$, from which we get the following result.

Theorem 3.2. *The base manifold (N^{2n-2}, J', g') of the above fibration is Kaehlerian if and only if*

$$d\Omega = du \wedge v - u \wedge dv$$

on M^{2n}.

Note also that by Proposition 2.6, $d\Omega = 0$ implies $du = dv = 0$ and hence $dF = 0$. Thus we have the following result.

Proposition 3.3. *If* M^{2n} *is Kaehlerian, then the base manifold* N^{2n-2} *is also Kaehlerian.*

4. Hermitian bicontact manifolds

We begin with the following elementary result on the topology of a compact bicontact manifold.

Theorem 4.1. *Let* M^{2n} *be a compact bicontact manifold, and let* $2p + 1$ *and* $2q + 1$ *denote the ranks of the bicontact forms* u *and* v *Then the betti numbers* b_{2p+1} *and* b_{2q+1} *are nonzero.*

Proof. As $(2p + 1) + (2q + 1) = 2n$ it suffices to show that b_{2p+1} is nonzero. We shall show that $u \wedge (du)^p$ has nonzero harmonic part. Suppose $u \wedge (du)^p$ has no harmonic part, then as $(du)^{p+1} = 0$, $u \wedge (du)^p$ is exact, say $d\alpha$. Now on a bicontact manifold $u \wedge (du)^p \wedge v \wedge (dv)^q$ is a volume element, hence, since $(dv)^{q+1} = 0$, we have

$$0 \neq \int_M u \wedge (du)^p \wedge v \wedge (dv)^q = \int_M d\alpha \wedge v \wedge (dv)^q = \int_M d(\alpha \wedge v \wedge (dv)^q) = 0 ,$$

a contradiction.

We shall now digress briefly to introduce the notion of a semi-invariant submanifold [3]. Let M^{2n} be an almost complex manifold with a vector field U and a 1-form u with $u(U) = 1$, and set $V = JU$, $v = -u \circ J$. Let $\iota : \bar{M} \to M^{2n}$ be a submanifold of M^{2n} such that 1) the transform of a vector tangent to $\bar{M}$ by J is in the space spanned by the tangent space of $\bar{M}$ and the vector U, 2) V is tangent to $\bar{M}$, and 3) $u \circ \iota_* = 0$; we then say that $\bar{M}$ is *semi-invariant with respect to* U. Note that U is never tangent to $\bar{M}$, for if it were, then $U = \iota_* \bar{U}$, and $1 = u(U) = u(\iota_* \bar{U}) = 0$, a contradiction.

Now define a tensor field φ of type (1, 1), a vector field ξ, and a 1-form η on $\bar{M}$ by

$$J\iota_* X = \iota_* \varphi X - \eta(X) U , \qquad V = \iota_* \xi .$$

We then have

$$-\iota_* X = \iota_* \varphi^2 X - \eta(\varphi X) U - \eta(X) \iota_* \xi ,$$

from which it follows that

$$\varphi^2 = -I + \eta \otimes \xi , \qquad \eta \circ \varphi = 0 .$$

Also

$$-U = JV = J\iota_*\xi = \iota_*\varphi\xi - \eta(\xi)U ,$$

giving

$$\varphi\xi = 0 , \qquad \eta(\xi) = 1 .$$

Thus we have the following result.

Proposition 4.2. *A submanifold of M^{2n}, which is semi-invariant with respect to U, admits an almost contact structure.*

Now computing $[J, J](\iota_*X, \iota_*Y)$ we have

$$\begin{aligned}[J, J](\iota_*X, \iota_*Y) &= \iota_*[\varphi, \varphi](X, Y) + d\eta(X, Y)\iota_*\xi - \eta(X)(\mathfrak{L}_U J)\iota_*Y \\ &\quad + \eta(Y)(\mathfrak{L}_U J)\iota_*X - ((\mathfrak{L}_{\varphi X}\eta)(Y) - (\mathfrak{L}_{\varphi Y}\eta)(X))U ,\end{aligned}$$

from which we obtain the following result.

Proposition 4.3. *If a submanifold is semi-invariant with respect to an analytic vector field U on a complex manifold M^{2n}, then its almost contact structure is normal.*

Returning to the bicontact case, we assume for the remainder of this section that M^{2n} is a Hermitian bicontact manifold as defined in § 2. We define a distribution $\mathscr{U}$ of dimension $2q + 1$ by

$$\mathscr{U} = \{X \mid i(X)u = 0, i(X)du = 0\} ,$$

where i denotes the interior product operator. We shall show that $\mathscr{U}$ is integrable. Let X and Y be vector fields belonging to $\mathscr{U}$. Then

$$0 = du(X, Y) = Xu(Y) - Yu(X) - u([X, Y]) = -u([X, Y]) .$$

Also for any Z

$$0 = du(X, Z) = Xu(Z) - u([X, Z]) = (\mathfrak{L}_X u)(Z) ,$$

and therefore

$$\begin{aligned}du([X, Y], Z) &= [X, Y]u(Z) - Zu([X, Y]) - u([[X, Y], Z]) \\ &= (\mathfrak{L}_{[X,Y]}u)(Z) = ((\mathfrak{L}_X\mathfrak{L}_Y - \mathfrak{L}_Y\mathfrak{L}_X)u)(Z) = 0 .\end{aligned}$$

Similarly the complementary distribution $\mathscr{V} = \{X \mid i(X)v = 0, i(X)dv = 0\}$ of dimension $2p + 1$ is integrable.

Theorem 4.4. *A Hermitian bicontact manifold M^{2n} with du of bidegree $(1, 1)$ is locally the product of two normal contact manifolds M^{2p+1} and M^{2q+1}.*

Proof. As noted above the distributions $\mathscr{U}$ and $\mathscr{V}$ are complementary and integrable. Thus M^{2n} is locally the product of the respective maximal integral submanifolds M^{2q+1} and M^{2p+1}. We shall show that the integral submanifold M^{2q+1} of $\mathscr{U}$ is semi-invariant with respect to U. Let $\iota\colon M^{2q+1} \to M^{2n}$ denote the immersion, and let X be tangent to M^{2q+1}, i.e., $\iota_* X \in \mathscr{U}$. Set $Y = J\iota_* X + v(\iota_* X)U$. Then

$$u(Y) = u(J\iota_* X) + v(\iota_* X) = -v(\iota_* X) + v(\iota_* X) = 0 ,$$

and

$$du(Y, Z) = du(J\iota_* X + v(\iota_* X)U, Z) = du(J\iota_* X, Z) = -du(\iota_* X, JZ) = 0$$

since du is of bidegree $(1, 1)$. Thus $Y \in \mathscr{U}$ so that M^{2q+1} is semi-invariant with respect to U, and hence by Proposition 4.3 its almost contact structure is normal. Finally as

$$\eta(X) = -g(J\iota_* X, U) = g(\iota_* X, V) = v(\iota_* X) ,$$

we have that $\eta \wedge (d\eta)^q \neq 0$ on M^{2q+1}. Similarly, M^{2p+1} is semi-invariant with respect to V, and is thus a normal contact manifold completing the proof.

Now let P and Q denote the projection maps to the tangent spaces of M^{2p+1} and M^{2q+1} respectively. We note for later use that $J(P - u \otimes U) = (P - u \otimes U)J$ as is easily verified, and hence that

$$JP = PJ + u \otimes V + v \otimes U .$$

We now compute the Lie derivative of P with respect to U and V. First note that

$$(\mathfrak{L}_U P)X = [U, PX] - P[U, X] .$$

Thus, if X is U or V, we clearly have $(\mathfrak{L}_U P)X = 0$. If X is orthogonal to U but also tangent to M^{2p+1}, then $PX = X$ and $[U, X]$ is again tangent to M^{2p+1} so that

$$(\mathfrak{L}_U P)X = [U, X] - [U, X] = 0 .$$

Finally, if X is orthogonal to V and tangent to M^{2q+1}, then $PX = 0$. Let Y be arbitrary. Then as U is Killing and P symmetric, we have

$$\begin{aligned} g((\mathfrak{L}_U P)X, Y) &= -g(P[U, X], Y) = -g(\nabla_U X, PY) + g(\nabla_X U, PY) \\ &= g(X, \nabla_U PY) - g(X, \nabla_{PY} U) = g(X, [U, PY]) = 0 . \end{aligned}$$

Similarly $\mathfrak{L}_V P = 0$, and thus P and $Q = I - P$ are projectable by the fibration of § 3.

On the base manifold N^{2n-2} of the fibration we define an almost product structure as follows.

$$P'X = \pi_* P\tilde{\pi} X\ , \qquad Q'X = \pi_* Q\tilde{\pi} X\ .$$

Then a direct computation shows that

$$P'^2 = P'\ ,\quad Q'^2 = Q'\ ,\quad P'Q' = Q'P' = 0\ ,\quad P' + Q' = I\ .$$

Moreover as both the distributions $\mathscr{U}$ and $\mathscr{V}$ are integrable, $[P, P] = 0$ so that

$$\begin{aligned}[P', P'](X, Y) &= \pi_* P^2\tilde{\pi}[\pi_*\tilde{\pi}X, \pi_*\tilde{\pi}Y] + [\pi_* P\tilde{\pi}X, \pi_* P\tilde{\pi}Y] \\ &\quad - \pi_* P\tilde{\pi}[\pi_* P\tilde{\pi}X, \pi_*\tilde{\pi}Y] - \pi_* P\tilde{\pi}[\pi_*\tilde{\pi}X, \pi_* P\tilde{\pi}Y] \\ &= \pi_*[P, P](\tilde{\pi}X, \tilde{\pi}Y) = 0\ .\end{aligned}$$

Thus the induced almost product structure on N^{2n-2} is integrable, and so N^{2n-2} is locally the product of two manifolds N^{2p} and N^{2q}.

We have already seen that J is projectable since U and V are analytic, and that $(J' = \pi_* J\tilde{\pi}, g')$ is a Hermitian structure on N^{2n-2}. Now let $\iota' : N^{2p} \to N^{2n-2}$ denote the immersion of N^{2p} in N^{2n-2}, and let X be a vector field on N^{2p}. Then using $JP = PJ + u \otimes V + v \otimes U$, we have

$$\begin{aligned}J'\iota'_* X &= \pi_* J\tilde{\pi}P'\iota'_* X = \pi_* JP\tilde{\pi}\iota'_* X = \pi_* PJ\tilde{\pi}\iota'_* X \\ &= \pi_* P\tilde{\pi}J'\iota'_* X = P'J'\iota'_* X\ ,\end{aligned}$$

and hence N^{2p} is an invariant submanifold of N^{2n-2} and consequently is a Hermitian submanifold of N^{2n-2}. Moreover, if N^{2n-2} is Kaehlerian, so is N^{2p} and similarly N^{2q}. Also, if each of the induced structures on N^{2p} and N^{2q} are Kaehlerian, so is the structure on N^{2n-2}. Thus using Theorems 3.1 and 4.4 and Proposition 3.2 we have

Theorem 4.5. *Let M^{2n} be a regular Hermitian bicontact manifold with du of bidegree $(1, 1)$. Then the base manifold N^{2n-2} of the induced fibration is locally the product of two Hermitian manifolds. Moreover, N^{2n-2} is locally the product of two Kaehler manifolds if and only if the fundamental 2-form Ω on M^{2n} satisfies $d\Omega = du \wedge v - u \wedge dv$.*

5. Curvature

In this section we give some results on the curvature of a Hermitian manifold admitting a holomorphic pair of automorphisms.

Proposition 5.1. *Let (M^{2n}, J, g) be a Hermitian manifold admitting a holomorphic pair of automorphisms $(U, V = JU)$. Then the sectional curvature of a section spanned by U and V vanishes.*

Proof. Since U is Killing, from $g(\nabla_V U, X) - g(\nabla_X U, V) = 0$ which was derived in the proof of Lemma 2.3 it follows that $2g(\nabla_V U, X) = 0$ and hence that $\nabla_V U = 0$. Moreover as U is a unit vector field, we have $0 = g(\nabla_X U, U) = -g(\nabla_U U, X)$ giving $\nabla_U U = 0$. Thus $g(R_{UV}U, V) = 0$, where R is the curvature tensor of g, and hence the sectional curvature of a section spanned by U and V vanishes.

Theorem 5.2. *If the Hermitian manifold M^{2n} of Theorem 3.1 has nonnegative sectional curvature, then the base manifold N^{2n-2} also has nonnegative curvature.*

Proof. First we note some relations.

$$[\tilde{\pi}X, \tilde{\pi}Y] = \tilde{\pi}[X, Y] + u([\tilde{\pi}X, \tilde{\pi}Y])U + v([\tilde{\pi}X, \tilde{\pi}Y])V .$$

Since U and V are Killing, we have

$$g(\nabla_{\tilde{\pi}X}\tilde{\pi}Y, U) = -g(\tilde{\pi}Y, \nabla_{\tilde{\pi}X}U) = -\tfrac{1}{2}du(\tilde{\pi}X, \tilde{\pi}Y) ,$$
$$g(\nabla_{\tilde{\pi}X}\tilde{\pi}Y, V) = -g(\tilde{\pi}Y, \nabla_{\tilde{\pi}X}V) = -\tfrac{1}{2}dv(\tilde{\pi}X, \tilde{\pi}Y) ,$$

and hence

$$\nabla_{\tilde{\pi}X}\tilde{\pi}Y = \tilde{\pi}\nabla'_XY - \tfrac{1}{2}du(\tilde{\pi}X, \tilde{\pi}Y)U - \tfrac{1}{2}dv(\tilde{\pi}X, \tilde{\pi}Y)V ,$$

where ∇' is the Riemannian connection of g'. Also, since $[U, \tilde{\pi}X]$ is vertical, $g(\nabla_U\tilde{\pi}X, \tilde{\pi}Y) = g(\nabla_{\tilde{\pi}X}U + [U, \tilde{\pi}X], \tilde{\pi}Y) = \frac{1}{2}du(\tilde{\pi}X, \tilde{\pi}Y)$, and similarly $g(\nabla_V\tilde{\pi}X, \tilde{\pi}Y) = \frac{1}{2}dv(\tilde{\pi}X, \tilde{\pi}Y)$.

We now compute the curvature.

$$\begin{aligned} g(R_{\tilde{\pi}X\tilde{\pi}Y}\tilde{\pi}X, \tilde{\pi}Y) &= g(\nabla_{\tilde{\pi}X}\nabla_{\tilde{\pi}Y}\tilde{\pi}X - \nabla_{\tilde{\pi}Y}\nabla_{\tilde{\pi}X}\tilde{\pi}X - \nabla_{[\tilde{\pi}X,\tilde{\pi}Y]}\tilde{\pi}X, \tilde{\pi}Y) \\ &= g(\nabla_{\tilde{\pi}X}(\tilde{\pi}\nabla'_YX - \tfrac{1}{2}du(\tilde{\pi}Y, \tilde{\pi}X)U - \tfrac{1}{2}dv(\tilde{\pi}Y, \tilde{\pi}X)V) \\ &\quad - \nabla_{\tilde{\pi}Y}\tilde{\pi}\nabla'_XX - \nabla_{[\tilde{\pi}X,\tilde{\pi}Y]}\tilde{\pi}X, \tilde{\pi}Y) \\ &= g(\tilde{\pi}\nabla'_X\nabla'_YX, \tilde{\pi}Y) - \tfrac{1}{2}du(\tilde{\pi}Y, \tilde{\pi}X)g(\nabla_{\tilde{\pi}X}U, \tilde{\pi}Y) \\ &\quad - \tfrac{1}{2}dv(\tilde{\pi}Y, \tilde{\pi}X)g(\nabla_{\tilde{\pi}X}V, \tilde{\pi}Y) - g(\tilde{\pi}\nabla'_Y\nabla'_XX, \tilde{\pi}Y) \\ &\quad - g(\tilde{\pi}\nabla'_{[X,Y]}X, \tilde{\pi}Y) - u([\tilde{\pi}X, \tilde{\pi}Y])g(\nabla_U\tilde{\pi}X, \tilde{\pi}Y) \\ &\quad - v([\tilde{\pi}X, \tilde{\pi}Y])g(\nabla_V\tilde{\pi}X, \tilde{\pi}Y) \\ &= g'(R'_{XY}X, Y)\circ\pi + \tfrac{3}{4}du(\tilde{\pi}X, \tilde{\pi}Y)^2 + \tfrac{3}{4}dv(\tilde{\pi}X, \tilde{\pi}Y)^2 \end{aligned}$$

since $du(\tilde{\pi}X, \tilde{\pi}Y) = \tilde{\pi}Xu(\tilde{\pi}Y) - \tilde{\pi}Yu(\tilde{\pi}X) - u([\tilde{\pi}X, \tilde{\pi}Y]) = -u([\tilde{\pi}X, \tilde{\pi}Y])$. Now for the sectional curvature K we have

$$K(\tilde{\pi}X, \tilde{\pi}Y) = \frac{-g(R_{\tilde{\pi}X\tilde{\pi}Y}\tilde{\pi}X, \tilde{\pi}Y)}{g(\tilde{\pi}X, \tilde{\pi}X)g(\tilde{\pi}Y, \tilde{\pi}Y) - g(\tilde{\pi}X, \tilde{\pi}Y)^2} .$$

Thus, if $K \geq 0$, then $g(R_{\tilde{\pi}X\tilde{\pi}Y}\tilde{\pi}X, \tilde{\pi}Y) \leq 0$ and hence

$$-g'(R'_{XY}X, Y)\circ\pi \geq \tfrac{3}{4}(du(\tilde{\pi}X, \tilde{\pi}Y)^2 + dv(\tilde{\pi}X, \tilde{\pi}Y)^2) ,$$

from which it follows that the sectional curvature $K'(X, Y) \geq 0$.

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